Chapter 8: Probability

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Section 8.1 Concepts of Probability

Introduction
The probability of a specified event is the chance or likelihood that it will occur. There are
several ways of viewing probability. One would be experimental in nature, where we
repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again
and it came up heads about half of the time; we would expect that in the future whenever we
flipped the coin it would turn up heads about half of the time. When a weather reporter says
“there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all
days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be subjective in nature, in other words an educated guess. If someone
asked you the probability that the Seattle Mariners would win their next baseball game, it
would be impossible to conduct an experiment where the same two teams played each other
repeatedly, each time with the same starting lineup and starting pitchers, each starting at the
same time of day on the same field under the precisely the same conditions. Since there are
so many variables to take into account, someone familiar with baseball and with the two
teams involved might make an educated guess that there is a 75% chance they will win the
game; that is, if the same two teams were to play each other repeatedly under identical
conditions, the Mariners would win about three out of every four games. But this is just a
guess, with no way to verify its accuracy, and depending upon how educated the educated
guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this
course we will mostly be concerned with theoretical probability, which is defined as
follows: Suppose there is a situation with \( n \) equally likely possible outcomes and that \( m \) of
those \( n \) outcomes correspond to a particular event; then the probability of that event is
defined as \( \frac{m}{n} \).

Basic Concepts
If you roll a die, pick a card from deck of playing cards, or randomly select a person and
observe their hair color, we are executing an experiment or procedure. In probability, we
look at the likelihood of different outcomes. We begin with some terminology.

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Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further.

The **sample space** is the set of all possible simple events.

**Example 1**

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: {1,2,3,4,5,6}

Some examples of simple events:
- We roll a 1
- We roll a 5

Some compound events:
- We roll a number bigger than 4
- We roll an even number

**Basic Probability**

Given that all outcomes are equally likely, we can compute the probability of an event \( E \) using this formula:

\[
P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally-likely outcomes}}\]

**Example 2**

If we roll a 6-sided die, calculate

a) \( P(\text{rolling a 1}) \)

b) \( P(\text{rolling a number bigger than 4}) \)

Recall that the sample space is \{1,2,3,4,5,6\}

a) There is one outcome corresponding to “rolling a 1”, so the probability is \( \frac{1}{6} \)

b) There are two outcomes bigger than a 4, so the probability is \( \frac{2}{6} = \frac{1}{3} \)

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.
Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is \( \frac{14}{20} = \frac{7}{10} \).

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now

1. At some random moment, you look at your clock and note the minutes reading.
   a. What is probability the minutes reading is 15?
   b. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

There are 52 cards in the deck and 4 Aces so \( P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769 \).

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.
Certain and Impossible events

An impossible event has a probability of 0.
A certain event has a probability of 1.
The probability of any event must be $0 \leq P(E) \leq 1$.

As you’re working through this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Complementary Events

Now let us examine the probability that an event does *not* happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with $n$ possible outcomes and an event $E$ that corresponds to $m$ of these outcomes. Then the remaining $n - m$ outcomes correspond to $E$ not happening, thus

$$P(\text{not } E) = \frac{n - m}{n} = \frac{n - m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The *complement* of an event is the event “$E$ doesn’t happen”.
The notation $\overline{E}$ is used for the complement of event $E$.
We can compute the probability of the complement using $P(\overline{E}) = 1 - P(E)$.
Notice also that $P(E) = 1 - P(\overline{E})$.

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.
The probability of *not* drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$
Sometimes you will see probabilities expressed as **odds**.

**Odds**

Odds of an event are typically expressed in the form $A:B$

$\left( \frac{\text{Number of outcomes corresponding to the event } E}{\text{Number of outcomes corresponding to } \overline{E}} \right)$

**Example 6**

If you pull a random card from a deck of playing cards, what are the odds is an Ace?

There are 4 Aces in the deck, and 48 cards that are not Aces, so the odds would be: 4:48, or 1:12

Notice how this is different than a probability – with probabilities we use the number of successes out of the total number of possible outcomes, while with odds we use the number of successes compared to the number of failures. As another example, if we flipped a coin, the odds of getting a heads would be 1:1.

**Probability of two independent events**

**Example 7**

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

We could list all possible outcomes:  $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$. Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

**Independent Events**

Events $A$ and $B$ are **independent events** if the probability of Event $B$ occurring is the same whether or not Event $A$ occurs.
Example 8

Are these events independent?

a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

c) You draw a card from a deck, then draw a second card without replacing the first.

a) The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

\[ P(A \text{ and } B) = P(A) \cdot P(B) \]

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.
Example 9

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

The probability of choosing a white pair of socks is \( \frac{6}{10} \).

The probability of choosing a white tee shirt is \( \frac{3}{7} \).

The probability of both being white is \( \frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35} \).

Example 10

The manufacturing process for a certain product has a 0.2% defect rate, meaning 2 products out of 1000 is defective on average. If two items are pulled randomly off the assembly line, what’s the probability both are defective?

The probability of each being defective is independent, so the probability of both defective is \( \frac{2}{1000} \cdot \frac{2}{1000} = \frac{4}{1,000,000} = \frac{1}{250,000} \).

Try it Now

2. A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of both events occurring. Now we will look at the probability of either event occurring.

Example 11

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin or a 6 on the die.

Here, there are still 12 possible outcomes: \{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\}

By simply counting, we can see that 7 of the outcomes have a head on the coin or a 6 on the die or both – we use or inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is \( \frac{7}{12} \). How could we have found this from the individual probabilities?
As we would expect, \( \frac{1}{2} \) of these outcomes have a head, and \( \frac{1}{6} \) of these outcomes have a 6 on the die. If we add these, \( \frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12} \), which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head and rolling a 6 is \( \frac{1}{12} \).

If we subtract out this double count, we have the correct probability: \( \frac{8}{12} - \frac{1}{12} = \frac{7}{12} \).

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

Example 12
Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

\[ P(\text{King or Queen}) = \frac{8}{52} \]

Note that in this case, there are no cards that are both a Queen and a King, so \( P(\text{King and Queen}) = 0 \). Using our probability rule, we could have said:

\[ P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} \]

In the last example, the events were mutually exclusive, so \( P(A \text{ or } B) = P(A) + P(B) \).

Example 13
Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so \( P(\text{red}) = \frac{26}{52} \)

There are four kings, so \( P(\text{King}) = \frac{4}{52} \)
There are two red kings, so \( P(\text{Red and King}) = \frac{2}{52} \)

We can then calculate

\[
P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}
\]

**Try it Now**

3. In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what is the probability at least one is white?

**Example 14**

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a red car *and* got a speeding ticket

b) Has a red car *or* got a speeding ticket.

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is \( \frac{15}{665} \approx 0.0226 \).

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is \( \frac{195}{665} \approx 0.2932 \).

We also could have found this probability by:

\[
P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket})
\]

\[
= \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}.
\]
Important Topics of this Section
- Experimental, subjective, and theoretic probability
- Events, sample space
- Basic probability
- Complementary events
- Odds
- Probability for independent events
- Computing an “or” probability

Try it Now Answers
1. There are 60 possible readings, from 00 to 59.  
   a. \( \frac{1}{60} \)  
   b. \( \frac{16}{60} \) (counting 00 through 15)

2. Since the second draw is made after replacing the first card, these events are independent.  
   The probability of an ace on each draw is \( \frac{4}{52} = \frac{1}{13} \), so the probability of an ace on both draws is \( \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169} \)

3. \( P(\text{white sock or white tee}) = \frac{6}{10} + \frac{3}{7} - \frac{9}{35} = \frac{27}{35} \)
Section 8.2 Conditional Probability and Bayes Theorem

Often it is required to compute the probability of an event given that another event has occurred. We call that conditional probability.

### Conditional Probability

The probability the event $B$ occurs, given that event $A$ has happened, is represented as $P(B \mid A)$.

This is read as “the probability of $B$ given $A$”

**Example 1**

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the conditional probability of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

$P(\text{ace on second draw} \mid \text{an ace on the first draw})$.

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw."

What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

**Example 2**

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.
Example 3

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket given they have a red car
b) Has a red car given they have a speeding ticket

<table>
<thead>
<tr>
<th></th>
<th>Speeding ticket</th>
<th>No speeding ticket</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red car</td>
<td>15</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>Not red car</td>
<td>45</td>
<td>470</td>
<td>515</td>
</tr>
<tr>
<td>Total</td>
<td>60</td>
<td>605</td>
<td>665</td>
</tr>
</tbody>
</table>

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

\[
P(\text{ticket} | \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1
\]

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

\[
P(\text{red car} | \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25.
\]

Notice from the last example that \(P(B | A)\) is not equal to \(P(A | B)\).

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events \(A\) and \(B\) are not independent, then

\[
P(A \text{ and } B) = P(A) \cdot P(B | A)
\]

Example 4

If you pull 2 cards out of a deck, what is the probability that both are spades?

The probability that the first card is a spade is \(\frac{13}{52}\).

The probability that the second card is a spade, given the first was a spade, is \(\frac{12}{51}\), since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is \(\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588\).
Example 5

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

You can satisfy this condition by having Case A or Case B, as follows: Case A) you can get the Ace of Diamonds first and then a black card or Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is \( \frac{1}{52} \). The probability that the second card is black given that the first card is the Ace of Diamonds is \( \frac{26}{51} \) because 26 of the remaining 51 cards are black. The probability is therefore \( \frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102} \).

Now for Case B: the probability that the first card is black is \( \frac{2}{52} = \frac{1}{2} \). The probability that the second card is the Ace of Diamonds given that the first card is black is \( \frac{1}{51} \). The probability of Case B is therefore \( \frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102} \), the same as the probability of Case 1.

Recall that the probability of A or B is \( P(A) + P(B) - P(A \text{ and } B) \). In this problem, \( P(A \text{ and } B) = 0 \) since the first card cannot be the Ace of Diamonds and be a black card.

Therefore, the probability of Case A or Case B is \( \frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51} \). The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is \( \frac{1}{51} \).

Try it Now

1. In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Example 6

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a) \( P(\text{not pregnant} | \text{positive test result}) \)

b) \( P(\text{positive test result} | \text{not pregnant}) \)
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<table>
<thead>
<tr>
<th></th>
<th>Positive test</th>
<th>Negative test</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pregnant</td>
<td>70</td>
<td>4</td>
<td>74</td>
</tr>
<tr>
<td>Not Pregnant</td>
<td>5</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>75</td>
<td>18</td>
<td>93</td>
</tr>
</tbody>
</table>

a) Since we know the test result was positive, we’re limited to the 75 women in the first column, of which 5 were not pregnant. \( P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067 \).

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. \( P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263 \).

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

Bayes Theorem

Bayes Theorem is a formulaic approach to complex conditional probability problems like the last example. However, using the formula is itself complicated, so we will focus on a more intuitive approach.

Example 7

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

**Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population).** The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: \( \frac{1}{1000} \). This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write \( P(\text{disease}) = 0.001 \).)
A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say \( P(\text{positive} \mid \text{disease}) = 1 \).)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that \( P(\text{positive} \mid \text{no disease}) = 0.05 \).)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute \( P(\text{disease} \mid \text{positive}) \). We already know that \( P(\text{positive} \mid \text{disease}) = 1 \), but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, 1/1000 of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect \((0.05)(999) = 49.95\) (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing \( P(\text{disease} \mid \text{positive}) \). There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

\[
P(\text{disease} \mid \text{positive}) \approx \frac{1}{51} \approx 0.0196
\]

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% do not have the disease.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and \((0.05)(99,900) = 4995\) test positive but do not have the disease, so the exact probability of having the disease if you test positive is

\[
P(\text{disease} \mid \text{positive}) \approx \frac{100}{5095} \approx 0.0196
\]

which is pretty much the same answer.
But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do not have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

**Bayes’ Theorem**

\[
P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A)P(B \mid \bar{A})}
\]

In our earlier example, this translates to

\[
P(\text{disease} \mid \text{positive}) = \frac{P(\text{disease})P(\text{positive} \mid \text{disease})}{P(\text{disease})P(\text{positive} \mid \text{disease}) + P(\text{no disease})P(\text{positive} \mid \text{no disease})}
\]

Plugging in the numbers gives

\[
P(\text{disease} \mid \text{positive}) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} \approx 0.0196
\]

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place
of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

**Example 8**

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9,800 who do not have the disease, 1% of them, or 98, will test positive.

<table>
<thead>
<tr>
<th></th>
<th>Positive test</th>
<th>Negative test</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have disease</td>
<td>180</td>
<td>20</td>
<td>200</td>
</tr>
<tr>
<td>Do not have disease</td>
<td>98</td>
<td>9,702</td>
<td>9,800</td>
</tr>
<tr>
<td>Total</td>
<td>278</td>
<td>9,822</td>
<td>10,000</td>
</tr>
</tbody>
</table>

So of the 278 total people who test positive, 180 will have the disease. Thus

\[ P(\text{disease} | \text{positive}) = \frac{180}{278} \approx 0.647 \]

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

\[ P(\text{disease} | \text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647 \]

**Example 9**

A company has found that 80% of its new management hires are meeting expectations, while 20% are not. Of the satisfactory hires, 75% had sales experience, while of the unsatisfactory hires, 55% had sales experience. What is the probability that a new hire with sales experience will meet expectations?

We can imagine 100 new hires. Of them, 80%, or 80, will meet expectations, and 20 will not. Of the 80 who meet expectations, 75%, or 60, had sales experience, and 20 did not. Of the 20 who did not meet expectations, 55%, or 11, had sales experience, and 9 did not.

Summarizing that in a table:
Now we can answer the question.

\[ P(\text{meet expectations | sales experience}) = \frac{60}{71} \approx 0.845 \]

So about 84.5% of new hires with sales experience will meet expectations.

### Try it Now

2. A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

### Important Topics of this Section

- Conditional probability
- Probability of “and” for conditional events
- Bayes Theorem

### Try it Now Answers

1. \[ \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3} \]

2. Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

\[ P(\text{disease | positive}) = \frac{500}{500 + 2985} = \frac{500}{3485} \approx 14.3\% \]
Section 8.3 Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting efficiently. When we get to the probability situations a bit later in this chapter we will need to count some very large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

<table>
<thead>
<tr>
<th>Appetizer</th>
<th>Main Course</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soup</td>
<td>Hamburger</td>
</tr>
<tr>
<td>Soup</td>
<td>Sandwich</td>
</tr>
<tr>
<td>Soup</td>
<td>Quiche</td>
</tr>
<tr>
<td>Salad</td>
<td>Hamburger</td>
</tr>
<tr>
<td>Salad</td>
<td>Sandwich</td>
</tr>
<tr>
<td>Salad</td>
<td>Quiche</td>
</tr>
<tr>
<td>Salad</td>
<td>Fajita</td>
</tr>
<tr>
<td>Salad</td>
<td>Pizza</td>
</tr>
<tr>
<td>Breadsticks</td>
<td>Hamburger</td>
</tr>
<tr>
<td>Breadsticks</td>
<td>Sandwich</td>
</tr>
<tr>
<td>Breadsticks</td>
<td>Fajita</td>
</tr>
<tr>
<td>Breadsticks</td>
<td>Pizza</td>
</tr>
</tbody>
</table>

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

<table>
<thead>
<tr>
<th></th>
<th>hamburger</th>
<th>sandwich</th>
<th>quiche</th>
<th>fajita</th>
<th>pizza</th>
</tr>
</thead>
<tbody>
<tr>
<td>soup</td>
<td>soup+burger</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>salad</td>
<td>salad+burger</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>bread</td>
<td>etc.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care what the possible meals are, only how many possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)
**Solution 3:** We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:

![Tree Diagram](image)

This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be to multiply the number of rows (3) by the number of columns (5) to get 15.
Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the basic counting rule:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are \( m \) items in the first category and \( n \) items in the second category, then the total number of available choices is \( m \cdot n \).

This is sometimes called the multiplication rule for probabilities.

Example 2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

There are 21 choices from the first category and 18 for the second, so there are \( 21 \cdot 18 = 378 \) possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are \( 3 \cdot 5 \cdot 2 = 30 \) possibilities.

Example 4

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in \( 2 \cdot 2 \cdot 2 = 8 \) different ways. Recall that another way to write \( 2 \cdot 2 \cdot 2 \) is \( 2^3 \), which is much more compact.
Try it Now

1. Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

**Permutations**

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

**Example 5**

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the same category (the category is: the letters of the word MATH) and each time we choose an item we do not replace it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code worth with the letters MATH.

In this example, we needed to calculate $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$.

**Factorial**

$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$

**Example 6**

How many ways can five different door prizes be distributed among five people?

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.
Example 7
A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the person who receives the $100 certificate, 24 remaining choices for the $25 certificate and 23 choices for the $10 certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example 8
Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded without replacement; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that the order of selection was important in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is not important since each of the three people receive the same prize. Situations where the order is not important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem without replacement where the order of selection is important. If we are arranging in order $r$ items out of $n$ possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.
Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing $r$ items out of $n$ possibilities without replacement and where the order of selection is important. In this situation we write:

**Permutations**

$$\binom{n}{r} = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

We say that there are $\binom{n}{r}$ permutations of size $r$ that may be selected from among $n$ choices without replacement when order matters.

It turns out that we can express this result more simply using factorials.

$$\binom{n}{r} = \frac{n!}{(n - r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

**Example 9**

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Since we are choosing 4 paintings out of 9 without replacement where the order of selection is important there are $9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

**Example 10**

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is $16P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$.

**Try it Now**

2. How many 5 character passwords can be made using the letters A through Z
   a. if repeats are allowed
   b. if no repeats are allowed
Combinations

In the previous section we considered the situation where we chose $r$ items out of $n$ possibilities without replacement and where the order of selection was important. We now consider a similar situation in which the order of selection is not important.

Example 11

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get $50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

\[
\text{ABC, ACB, BAC, BCA, CAB, CBA}
\]

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted six times. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $13,800/6 = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items without replacement where the order of selection is not important. If we are choosing $r$ items out of $n$ possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by $\frac{n!}{(n-r)!}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing $r$ items out of $n$ possibilities without replacement where the order of selection is not important.
Combinations

\[ nCr = \frac{n!}{r!(n-r)!} \]

We say that there are \( nCr \) combinations of size \( r \) that may be selected from among \( n \) choices without replacement where order doesn’t matter.

We can also write the combinations formula in terms of factorials:

\[ nCr = \frac{n!}{(n-r)!r!} \]

Example 12
A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Since we are choosing 4 people out of 35 without replacement where the order of selection is not important there are \( 35C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360 \) combinations.

Try it Now
3. The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 13
The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are \( 15C_{10} = 3003 \) ways to choose the 10 Republicans and \( 14C_9 = 2002 \) ways to choose the 9 Democrats. But now what? How do we finish the problem?
Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

**Probability using Permutations and Combinations**

We can use permutations and combinations to help us answer more complex probability questions.

**Example 14**

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation $10P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $
\frac{10P_4}{10^4} = \frac{5040}{10000} = 0.504$

**Example 15**

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000$. In this lottery, the order the numbers are drawn in doesn’t matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player’s ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is $48C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player’s ticket, so the probability of winning the grand prize is:

$$\frac{6C_6}{48C_6} = \frac{1}{12271512} \approx 0.0000000815$$
Example 16

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is \( 48 \text{C}_6 = 12,271,512 \). In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by \( 6 \text{C}_5 = 6 \) and the number of ways to choose 1 out of the 42 losing numbers is given by \( 42 \text{C}_1 = 42 \). Thus the number of favorable outcomes is then given by the Basic Counting Rule: \( 6 \text{C}_5 \cdot 42 \text{C}_1 = 6 \cdot 42 = 252 \). So the probability of winning the second prize is:

\[
\frac{6 \text{C}_5 \cdot 42 \text{C}_1}{48 \text{C}_6} = \frac{252}{12271512} \approx 0.0000205
\]

Try it Now

4. A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Example 17

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, \( 52 \text{C}_5 \). This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be \( 4 \text{C}_1 \) ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be \( 48 \text{C}_4 \) ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be \( 4 \text{C}_1 \cdot 48 \text{C}_4 \) ways to choose one ace and four non-Aces.

Putting this all together, we have:

\[
P(\text{one Ace}) = \frac{4 \text{C}_1 \cdot 48 \text{C}_4}{52 \text{C}_5} = \frac{778320}{2598960} \approx 0.299
\]
Example 18

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

\[
P(\text{two Aces}) = \frac{\binom{4}{2} \binom{48}{3}}{\binom{52}{5}} = \frac{103776}{2598960} \approx 0.0399
\]

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now

5. Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 19

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves “What is the alternative to having at least one shared birthday?” In this case, the alternative is that there are no shared birthdays. In other words, the alternative to “at least one” is having none. In other words, since this is a complementary event,

\[
P(\text{at least one}) = 1 - P(\text{none})
\]

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's
ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is $364/365$. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is $363/365$.

We want the second person not to share a birthday with you and the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$ 

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

**Example 20**

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{365 \cdot P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

**Example 21**

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{365 \cdot P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!
If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: http://statweb.stanford.edu/~susan/surprise/Birthday.html, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now

6. Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Important Topics of this Section

Basic counting techniques
Basic counting rule
Factorial
Permutations
Combinations
Probability using Permutations and Combinations

Try it Now Answers

1. \(8 \cdot 11 \cdot 5 = 440\) menu combinations

2. There are 26 characters.  
   a. \(26^5 = 11,881,376\).  
   b. \(26P_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600\)

3. Order does not matter. \(29C_{19} = 20,030,010\) possible subcommittees

4. There are \(5^{10} = 9,765,625\) different ways the exam can be answered. There are 9 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 36 ways the test could be answered with one wrong answer.  
   \[P(9 \text{ answers correct}) = \frac{36}{5^{10}} \approx 0.0000037\]  
   chance

5. \[P(\text{three Aces and two Kings}) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \frac{24}{2598960} \approx 0.0000092\]

6. \[P(\text{shared birthday}) = 1 - \frac{365 \cdot P_{10}}{365^{10}} \approx 0.117\]
Section 8.4 Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 1

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets $1 on a single number. If that number is spun on the wheel, then they receive $36 (their original $1 + $35). Otherwise, they lose their $1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Suppose you bet $1 on each of the 38 spaces on the wheel, for a total of $38 bet. When the winning number is spun, you are paid $36 on that number. While you won on that one number, overall you’ve lost $2. On a per-space basis, you have “won” -$2/$38 ≈ -0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose $1 and a very few people (about 1 person out of every 38) gain $35 (the $36 they win minus the $1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability of outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$35</td>
<td>$\frac{1}{38}$</td>
</tr>
<tr>
<td>-$1</td>
<td>$\frac{37}{38}$</td>
</tr>
</tbody>
</table>

Notice that if we multiply each outcome by its corresponding probability we get $35 \cdot \frac{1}{38} = 0.9211$ and $-1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

1 Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/
Expected Value

**Expected Value** is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

**Try it Now**

1. You purchase a raffle ticket to help out a charity. The raffle ticket costs $5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth $4000. Compute the expected value for this raffle.

**Example 2**

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. If they match 5 numbers, then win $1,000. It costs $1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\binom{6}{6}\binom{48}{6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\binom{6}{5}\binom{42}{1}\binom{48}{6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability of outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$999,999</td>
<td>\frac{1}{12271512}</td>
</tr>
<tr>
<td>$999</td>
<td>\frac{252}{12271512}</td>
</tr>
<tr>
<td>-$1</td>
<td>$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$</td>
</tr>
</tbody>
</table>

The expected value, then is:

$$\left(\$999,999\right)\frac{1}{12271512} + \left(\$999\right)\frac{252}{12271512} + \left(-\$1\right)\frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.
In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the average winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

### Try it Now

2. A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

#### Example 3

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year. An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability of outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100,000 - $275 = $99,725</td>
<td>0.00242</td>
</tr>
<tr>
<td>-$275</td>
<td>1 – 0.00242 = 0.99758</td>
</tr>
</tbody>
</table>

The expected value is $(99,725)(0.00242) + (-$275)(0.99758) = -$33.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

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Expected value is also used by businesses for decision making.

Example 4
A company is considering two acquisitions. They have evaluated the potential future potential of each. Acquisition A has a 30% probability of increasing profit by $20MM, a 60% probability of increasing profit by $5MM, and a 10% probability of decreasing profit by $5MM. Acquisition B has a 10% probability of increasing profit by $40MM, a 50% probability of increasing profit by $10MM, and a 40% probability of decreasing profit by $4MM. Which acquisition is more prudent?

To compare these, we can look at the expected value:
Acquisition A: \( (20)(0.30) + (5)(0.60) + (-5)(0.10) = $8.5MM \)
Acquisition B: \( (40)(0.10) + (10)(0.50) + (-4)(0.40) = $7.4MM \)

Since Acquisition A has a higher expected value, it is the more prudent acquisition.

Important Topics of this Section
- Expected value
- Fair game

Try it Now Answers
1. \( \left( \frac{3995}{2000} \right) + \left( -\frac{5}{2000} \right) \cdot \frac{1999}{2000} \approx -$3.00 \)